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Fiscal Policy and Indeterminacy in Models of Endogenous Growth*

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1 Introduction

In this paper we investigate the dynamic behavior of the two-sector endogenous growth model. We demonstrate that indeterminacy—continua of rational expectations equilibria consistent with self-fulfilling prophecies—emerges when we introduce sector-specific external effects and distortionary taxation in tandem.

In his 1988 paper, Lucas [20] argued that one-sector growth models fail to account for the wide variation in observed cross-country growth rates because they do not endogenize the accumulation of labor augmenting technology. Building on earlier work by Uzawa [33], Lucas studied the steady-state growth properties of a model that included an extra sector producing human capital. Lucas proposed that beyond the direct role of human capital in the production of final goods, higher levels of education, or increased investment in research and development, generate indirect improvements in the overall efficiency of production—external economies not captured at the level of the individual worker or firm. Subsequent work by Benhabib and Perli [4] and Xie [36] demonstrated that the presence of these economy-wide human capital externalities is sufficient to generate indeterminacy in the Lucas model, if the intertemporal elasticity of substitution is very high. In Sections 2–4 we present an endogenous growth model with factor taxation and government expenditure. We limit the scope of external effects to be sector-specific; only the portion of human capital employed in a sector generates spillover effects in that sector.

Schmitt-Grohé and Uribe [32] find that equilibria are indeterminate in the one sector real business cycle model if the tax rate on labor is sufficiently high, and if government expenditure is not too pro-cyclical. Guo and Lansing [16] demonstrate that in a one sector model with increasing returns, the dynamic system undergoes a Hopf bifurcation if capital accumulation is subsidized. In Section 5 we demonstrate that in the two-sector endogenous growth model with sector-specific external effects, Hopf bifurcations and indeterminacy emerge with positive rates of capital taxation, and plausible intertemporal elasticities of substitution, if production is convex in effective labor.

In each of the last two sections, we explore a generalization of the model. In both sections, we find that empirically plausible tax policies can induce indeterminacy when production is non-convex in effective labor, or even when increasing returns are completely absent. In Section 6 we follow Benhabib and Perli [4], Ladron-de-Guevara *et al.* [19], and Mino [23], [25], and introduce a labor-leisure choice—when labor supply is elastic, indeterminacy is consistent with downward sloping labor demand. If the tax on capital

income is low, two balanced growth paths emerge—one determinate, with a high rate of steady state growth, and the other indeterminate with low growth.

In the final section, we introduce physical capital as an input in the human capital sector. Bond *et al.* [5] demonstrate that high capital taxation is sufficient for equilibria to be indeterminate in a generalized two-sector model, and Mino [24] finds that sector specific external effects will induce similar results. Combining the two, we find that for a wide range of fiscal policies, indeterminacy emerges when social returns in both sectors are constant, and only a small degree of external effects are present in the human capital producing sector.

2 The Basic Model

Romer [30] was the first to consider the implications of including external effects in an endogenous growth model. Lucas [20] followed by introducing external effects into the two-sector model developed by Uzawa [33] in the early 1960's. Although neither increasing returns or external effects are necessary for the existence of endogenous growth, the inclusion of human capital externalities enables Lucas' model to accommodate two important observations: there are large differences in the rental rates for human capital (the wage for a given level of skill) across countries, and also differences between the growth rates of physical and human capital within countries.

In contrast to Romer, Lucas eliminated scale effects from his model by restricting the external effects to the average, rather than the aggregate amount of human capital. Nonetheless, when adapting the external effects from Romer's one sector model to a model with two sectors, Lucas maintained the assumption that the economy's entire stock of human capital directly increases productivity in the final goods sector. Therefore in Lucas' model there are two distinct types of external effects—positive spillovers across firms within the final goods sector, and also positive spillovers from the human capital employed in the human capital sector, that accrue to firms in the production sector. In this paper, we restrict external effects to be sector-specific—we eliminate the spillovers between sectors.

There are several reasons to prefer this more modest specification of human capital externalities. First, the inclusion of spillovers between sectors is not necessary to generate either differential rates of steady state growth for the two types capital, or higher rental rates for human capital in rich countries—as we demonstrate below, the inclusion of sector specific

externalities is sufficient. Second, the most obvious spillovers result from complementarities between the skills of workers—personnel in a sector interact and learn from each other. Finally, although increases in the total stock of knowledge certainly enhance productivity—this stock is produced in both domestic and foreign human capital sectors. By restricting spillovers to the portion of human capital employed in each sector, we focus our attention on the endogenous, domestically produced portion of human capital.

A number of empirical studies find evidence of sector specific external effects and increasing returns to scale. Paul and Siegel [26] find that sizeable increasing returns are prevalent in U.S. manufacturing, and that at least two-thirds of the increasing returns can be ascribed to agglomeration effects—sector specific externalities at the two-digit industry level. Harrison [18] finds evidence of increasing returns but rejects spillovers between sectors, and Benhabib and Jovanovic [3], demonstrate that the source of aggregate increasing returns to scale are not the external effects from physical capital inputs. Taken together the evidence suggests that increasing returns exist—generated by sector-specific external effects from inputs other than physical capital.

The economy is composed of a government sector and a large number of households—their behavior is represented by the intertemporal maximization of an infinite-lived representative consumer. The consumer chooses the dynamic paths of c , consumption, and $u \in (0, 1)$, the fraction of time or human capital devoted to work in the final goods sector:

$$\max_c \int_0^\infty e^{-\rho t} \frac{c^{1-\sigma}}{1-\sigma} dt, \quad (\text{P.1})$$

subject to the constraints:

$$\dot{k} = (1 - \tau_l) w u h + (1 - \tau_k) r k - c$$

$$\dot{h} = \nu (1 - u)^{1-\gamma} h^{1-\gamma} (1 - u_a)^\gamma h_a^\gamma,$$

where k is the individual's stock of physical capital, h his stock of human capital, r and w the rental rates of physical and human capital, τ_l and τ_k , the tax rates on labor and capital income, $\sigma > 0$, the inverse of the intertemporal elasticity of substitution, and ρ , a positive discount rate. Time not devoted to work for wages is spent accumulating human capital— ν is the maximum rate at which human capital can be accumulated. The terms u_a and h_a are respectively, the average fraction of hours devoted to work in the final goods sector, and the per-capita stock of human capital. The parameter γ regulates the size of the external effect in the human capital sector, and the degree of that sector's internal decreasing returns.

Physical goods are produced by a combination of physical capital and effective labor:

$$y = (u_a h_a)^\beta F(k, uh). \quad (1)$$

The term $(u_a h_a)^\beta$ captures the efficiency enhancing external effects of that portion of the human capital stock employed in the final goods sector, just as the term $(1 - u_a)^\gamma h_a^\gamma$ is the analogous external effect for the sector producing human capital.² We assume $F : R^2 \rightarrow R$ is homogenous of degree one, so the degree of increasing returns to scale at the social level is governed by the magnitude of $\beta \geq 0$. Internal factor returns will be:

$$r = (u_a h_a)^\beta F_k(k, uh), \quad (2)$$

$$w = (u_a h_a)^\beta F_\phi(k, uh). \quad (3)$$

We define $\phi = uh$, as effective labor and assume the Cobb-Douglas form $F(k, \phi) = k^\alpha \phi^{1-\alpha}$ for the production function. The parameter space is Θ : $\theta \equiv (\alpha, \beta, \gamma, \nu, \rho, \sigma)$, and $\theta \in \Theta$, where $\Theta = (0, 1) \times R_+^2 \times R_{++}^3$. If λ and μ are the costate variables for physical and human capital the first order necessary conditions for an interior solution are:

$$\frac{e^{-\rho t}}{c^\sigma} = \lambda \quad (4)$$

$$(1 - \alpha)(1 - \tau_l) \lambda h (u_a h_a)^\beta k^\alpha (uh)^{-\alpha} = (1 - \gamma) \nu \mu (1 - u)^{-\gamma} h^{1-\gamma} (1 - u_a)^\gamma h_a^\gamma \quad (5)$$

$$\alpha \lambda (1 - \tau_k) (u_a h_a)^\beta k^{\alpha-1} (uh)^{1-\alpha} = -\dot{\lambda} \quad (6)$$

$$\mu \nu (1 - \gamma) (1 - u)^{1-\gamma} h^{-\gamma} (1 - u_a)^\gamma h_a^\gamma + \lambda (1 - \alpha) (1 - \tau_l) (u_a h_a)^\beta k^\alpha (uh)^{-\alpha} u = -\dot{\mu}, \quad (7)$$

plus the two transversality conditions: $\lim_{t \rightarrow \infty} \lambda k = 0$, and $\lim_{t \rightarrow \infty} \mu h = 0$.

We assume that the government sector's budget is always balanced:

$$g k^\alpha \phi^{1-\alpha+\beta} = \tau_l w \phi + \tau_k r k. \quad (8)$$

where g is the fraction of output consumed by the government.³ The dynamic behavior of the economy is described by the laws of motion for per-capita consumption, physical capital, and effective labor:

$$\dot{c} = \frac{1}{\sigma} \left(\alpha (1 - \tau_k) k^{\alpha-1} \phi^{1-\alpha+\beta} - \rho \right) c \quad (9)$$

$$\dot{k} = (1 - g) k^\alpha \phi^{1-\alpha+\beta} - c. \quad (10)$$

$$\dot{\phi} = \frac{\alpha}{\alpha - \beta} \left(\frac{(1 - \gamma) \nu}{\alpha} - \frac{c}{k} + (\tau_k - g) k^{\alpha-1} \phi^{1-\alpha+\beta} \right) \phi. \quad (11)$$

3 The Reduced Model and Balanced Growth

Ordinarily, the dynamic behavior of the Lucas model is described by the laws of motion of four variables: hours worked in the final goods sector, consumption, and the stocks of human and physical capital. If external effects are sector-specific, hours worked and human capital can be combined into one variable, effective labor, and two laws of motion in stationary consumption and physical capital describe the evolution of the economy.

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{1}{\sigma} \left((1 - \tau_k) \alpha \tilde{k}^{\alpha-1} - \rho \right) - \vartheta \left(\frac{(1 - \gamma) \nu}{\alpha} - \frac{\tilde{c}}{\tilde{k}} + (\tau_k - g) \tilde{k}^{\alpha-1} \right) \quad (12)$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = (1 - g) \tilde{k}^{\alpha-1} - \frac{\tilde{c}}{\tilde{k}} - \vartheta \left(\frac{(1 - \gamma) \nu}{\alpha} - \frac{\tilde{c}}{\tilde{k}} + (\tau_k - g) \tilde{k}^{\alpha-1} \right), \quad (13)$$

where $\tilde{c} = c\phi^{-\frac{1-\alpha+\beta}{1-\alpha}}$, $\tilde{k} = k\phi^{-\frac{1-\alpha+\beta}{1-\alpha}}$, and $\vartheta = \frac{1-\alpha+\beta}{\alpha-\beta} \frac{\alpha}{1-\alpha}$.

The term τ_l is absent from both laws of motion—without the zero deficit assumption (8), temporary fluctuations in the tax on wages do not affect the economy. This tax is equivalent to a lump-sum tax, net of transfer payments.⁴ The tax rate on capital income and government spending do appear in the laws of motion—they do affect the behavior of the economy, including the value of u along the transition path. However, neither the share of government expenditure in output, nor any tax rate, affects the allocation of time between the two sectors once the economy has converged to the balanced growth path:

$$u^* = \frac{\rho - (\eta - \gamma) \nu}{(1 - \eta) \nu}, \quad (14)$$

where $\eta = \frac{(1-\sigma)(1-\alpha+\beta)}{(1-\alpha)}$ is the product of the curvature of the utility function, and the ratio between the social marginal product of human capital, and its internal marginal product.

For an interior solution to P.1 to exist, agents cannot be so impatient they allocate all available time to immediate production, or so patient they postpone all labor market activity to maximize the accumulation of human capital. Bounds on the discount rate ensure that the steady state fraction of hours devoted to work in the final goods sector falls within the unit interval:

$$\begin{aligned} \Theta_1 &\equiv \{ \theta \in \Theta \mid (\eta - \gamma) \nu < \rho < (1 - \gamma) \nu \text{ and } \eta < 1 \} \\ \Theta_2 &\equiv \{ \theta \in \Theta \mid (1 - \gamma) \nu < \rho < (\eta - \gamma) \nu \text{ and } \eta > 1 \}, \end{aligned} \quad (15)$$

where $\Theta_1 \cup \Theta_2 \subseteq \Theta$. If $\theta \in \Theta_1 \cup \Theta_2$, then $u^* \in (0, 1)$.

Setting (12) and (13) equal to zero, we solve for steady state consumption and capital:

$$\tilde{c}^* = \frac{(\alpha - \beta)}{\alpha} \left(\Sigma_1 - \Sigma_2 \frac{\tau_k - g}{1 - \tau_k} \right) \tilde{k}^* \quad (16)$$

$$\tilde{k}^* = \left(\frac{\alpha(1 - \eta)(1 - \alpha)(1 - \tau_k)}{(1 - \gamma)(1 - \alpha + \beta)\nu\sigma - \beta\rho} \right)^{\frac{1}{1 - \alpha}}, \quad (17)$$

where $\Sigma_1 = \nu \left(\frac{\rho - (1 - \gamma)}{(1 - \eta)} + \frac{1 - \gamma}{\alpha - \beta} \right)$ and $\Sigma_2 = \frac{\beta\rho - (1 - \alpha + \beta)(1 - \gamma)\nu\sigma}{(1 - \alpha)(\alpha - \beta)(1 - \eta)}$.

Proposition 1 *There exists a unique interior balanced growth path if and only if $\theta \in \Theta_1 \cup \Theta_2$ and $\frac{\Sigma_1}{\Sigma_2} < \frac{\tau_k - g}{1 - \tau_k}$.*

Proof. If $\theta \in \Theta_1$ [$\theta \in \Theta_2$] then $\eta < 1$ [$\eta > 1$], the numerator of (17) is positive [negative], $\sigma(1 - \alpha + \beta) > \beta$, [$\sigma(1 - \alpha + \beta) < \beta$] and $(1 - \gamma)\nu > \rho$ [$(1 - \gamma)\nu < \rho$]. Therefore $(1 - \gamma)(1 - \alpha + \beta)\nu\sigma - \beta\rho > 0$ [$(1 - \gamma)(1 - \alpha + \beta)\nu\sigma - \beta\rho < 0$], the denominator of (17) is positive [negative] as well and \tilde{k}^* is positive and unique. If $\theta \in \Theta_1 \cup \Theta_2$ and $\alpha > \beta$ [$\alpha < \beta$] then $\Sigma_2 < 0$ [$\Sigma_2 > 0$]. Therefore if $\theta \in \Theta_1 \cup \Theta_2$ and $\frac{\Sigma_1}{\Sigma_2} < \frac{\tau_k - g}{1 - \tau_k}$ then $(\alpha - \beta)\Sigma_1 > (\alpha - \beta)\Sigma_2 \frac{\tau_k - g}{1 - \tau_k}$ and \tilde{c}^* in (16) is positive and unique. ■

If $\theta \in \Theta_1 \cup \Theta_2$ and $\tau_k = \tau_l = g$, a unique interior balanced growth path exists if $\alpha < \beta$, or if $\alpha > \beta$ and $\gamma < 1 - \alpha + \beta$. We also define the set $\Theta_3 \subseteq \Theta$:

$$\Theta_3 \equiv \{\theta \in \Theta | \rho = (1 - \gamma)\nu \text{ and } \eta = 1\}. \quad (18)$$

If $\theta \in \Theta_3$ the numerators and denominators in (16) and (17) are equal to zero, implying the existence of an infinite number of balanced growth paths.

Along the balanced growth path, physical output, consumption, wages, and physical capital, grow at the rate:

$$\kappa = \frac{(1 - \alpha + \beta)((1 - \gamma)\nu - \rho)}{(1 - \alpha)(1 - \eta)}. \quad (19)$$

The steady state growth rate of human capital is $\frac{(1 - \gamma)\nu - \rho}{1 - \eta}$ —the presence of sector-specific externalities is sufficient to ensure that physical capital accumulates faster than human capital. Along the balanced growth path, the rental rate of human capital grows at the rate $\frac{\beta((1 - \gamma)\nu - \rho)}{(1 - \alpha)(1 - \eta)}$. If $\beta > 0$, the wage for a given skill-level is directly related to the per-capita stock of human capital. Although we eliminated inter-sector spillovers, wages for a given level of skill, just as in Lucas [20], are highest in the most developed countries.

4 Optimal Taxation

Generally, in an economy without external effects, the optimal long-run tax on capital is zero. The entire burden for the finance of a fixed amount of government expenditure falls on labor—the factor which is inelastically supplied. However, if the government is consuming a fixed, positive portion of output, the optimal long-run tax on capital is positive.

Proposition 2 *If external effects are sector-specific, and the production function is concave in both inputs at the social level, the Ramsey optimal fiscal policy equalizes taxes between capital and labor income.*

Proof. Consider the maximization problem of a social planner who chooses the tax on capital to maximize utility, subject to the equilibrium conditions (9), (10) and (11). The current value Hamiltonian will be:

$$H(c, \phi, k, \tau_k, \omega, \zeta, \psi) = \frac{c^{1-\sigma}}{1-\sigma} + \omega \frac{1}{\sigma} \left[(1-\tau_k) \alpha k^{\alpha-1} \phi^{1-\alpha+\beta} - \rho \right] c \quad (\text{H.1})$$

$$+ \zeta \left[(1-g) k^\alpha \phi^{1-\alpha+\beta} - c \right] + \psi \frac{\alpha}{\alpha-\beta} \left[\frac{(1-\gamma)\nu}{\alpha} - \frac{c}{k} + (\tau_k - g) k^{\alpha-1} \phi^{1-\alpha+\beta} \right] \phi,$$

where ω , ζ , and ψ are the costate variables that correspond to each of the incentive compatibility constraints. Differentiating H.1 with respect to ϕ and τ_k :

$$\frac{\partial H}{\partial \phi} = (1-\alpha+\beta) \left((1-\tau_k) \frac{\alpha c}{\sigma k} \omega + (1-g) \zeta \right) k^\alpha \phi^{-\alpha+\beta} \quad (20)$$

$$+ \left[\frac{\alpha}{\alpha-\beta} \left(\frac{(1-\gamma)\nu}{\alpha} - \frac{c}{k} \right) + (\tau_k - g) \left(\frac{\alpha}{\alpha-\beta} + \vartheta(1-\alpha) \right) k^{\alpha-1} \phi^{1-\alpha+\beta} \right] \psi = \rho \psi - \dot{\psi}$$

$$\frac{\partial H}{\partial \tau_k} = -\omega \frac{1}{\sigma} c + \psi \frac{1}{\alpha-\beta} \phi = 0. \quad (21)$$

We differentiate (21) with respect to time, and combine with (20), to replace ψ and $\dot{\psi}$:

$$\omega \left((1-\tau_k) + \frac{1}{\phi} (\tau_k - g) \right) c \tilde{k}^{\alpha-1} + \zeta \frac{\sigma}{\alpha} (1-g) \tilde{k}^\alpha \phi^{\frac{1-\alpha+\beta}{1-\alpha}} = \frac{\rho \omega c - \dot{\omega} c - \omega \dot{c}}{\vartheta(1-\alpha)}. \quad (22)$$

Inserting (9) into (22), rewriting in terms of stationary variables, and replacing the terms for stationary consumption and capital with their balanced growth values yields the condition:

$$\frac{\sigma(1-\alpha)\vartheta\zeta}{\alpha\tilde{c}^*} \frac{1}{\omega} \left(\frac{1}{\sigma} \left((1-\tau_k) \alpha \tilde{k}^{*\alpha-1} - \rho \right) + \frac{\tilde{c}^*}{\tilde{k}^*} \right) + \frac{\dot{\omega}}{\omega} \quad (23)$$

$$+ (1-\tau_k) \left((1-\alpha)\vartheta + \frac{1}{\sigma} \alpha \right) \tilde{k}^{*\alpha-1} - \left(1 + \frac{1}{\sigma} \right) \rho = (1-\alpha)\vartheta \frac{\tilde{k}^{*\alpha-1}}{\phi} (g - \tau_k).$$

Along the balanced growth path, the costate variables ζ and ω must grow at the same constant rate. Therefore $\frac{\zeta}{\omega}$ and $\frac{\dot{\omega}}{\omega}$ are constant, and the first two terms on the left-hand

side of (23) are constant as well. The other terms on the left-hand side are defined in terms of stationary variables and must be constant. The right hand side is divided by the term ϕ which is non-stationary—the equality is only maintained for every period if $\tau_k = g$.

Sufficient conditions for a maximum can be derived by employing the Arrow Theorem. Define the maximized Hamiltonian $H^0(c, \phi, k, \omega, \zeta, \psi) = \max_{\tau_k} H(c, \phi, k, \tau_k, \omega, \zeta, \psi)$:

$$\begin{aligned} H^0(c, \phi, k, \omega, \zeta, \psi) = & \frac{c^{1-\sigma}}{1-\sigma} + \omega \frac{1}{\sigma} \left[(1-g) \alpha k^{\alpha-1} \phi^{1-\alpha+\beta} - \rho \right] c \\ & + \zeta \left[(1-g) k^\alpha \phi^{1-\alpha+\beta} - c \right] + \psi \frac{\alpha}{\alpha-\beta} \left[\frac{(1-\gamma)\nu}{\alpha} - \frac{c}{k} \right] \phi. \end{aligned} \quad (\text{H.2})$$

We write $\frac{\partial^2 H^0}{\partial \phi \partial \phi}$ in terms of the stationary co-state variables $\tilde{\omega} = \omega \phi^{-\frac{1-\alpha+\beta}{1-\alpha}}$, $\tilde{\zeta} = \zeta \phi^{-\frac{1-\alpha+\beta}{1-\alpha}}$:

$$\frac{\partial^2 H^0}{\partial \phi \partial \phi} = (1-\alpha+\beta)(\beta-\alpha)(1-g) \left[\frac{\alpha}{\sigma} \tilde{\omega} \tilde{c} + \tilde{\zeta} k \right] \tilde{k}^{\alpha-1} \phi^{4+\frac{2\beta}{1-\alpha}}, \quad (24)$$

which is negative if and only if $\alpha > \beta$ (if the production function is concave so is H.1).⁵ Finally, from the government's budget constraint (8), the tax on wages τ_l , is also g . ■

In this economy, government expenditure is analogous to a tax on output, and the second best optimum is achieved by equalizing all the different taxes rates. In fact, as we demonstrate below, when this policy is adopted the dynamic behavior of the model mimics the behavior of a model with no government consumption.

Chamley [8] found that in Lucas' model the optimal tax rate on capital income increases with the magnitude of the external effects in the total stock of human capital. Proposition 2 demonstrates that the size of sector-specific externalities do not change optimal policy. Only spillover effects between sectors justify tax rates on capital income higher than g —and it is only the inefficiencies associated with that subset of the total external effects in Chamley [8], or Lucas [20], that a higher tax on capital income can ameliorate.

5 Taxes and Stability of the Balanced Growth Path

To find the local stability properties of the reduced system in the neighborhood of the unique balanced growth path, we linearize the laws of motion (12) and (13) around the steady state. The Jacobian \mathbf{J}^B , of the linearized system, can be expressed as the sum of two matrices— \mathbf{J}^A , the Jacobian of a model without a government sector, and a second matrix, \mathbf{M} , multiplied by $\frac{\tau_k - g}{1 - \tau_k}$ which represents the effects of fiscal policy:

$$\mathbf{J}^B = \mathbf{J}^A + \frac{\tau_k - g}{1 - \tau_k} \mathbf{M}, \quad (25)$$

where:

$$\mathbf{J}^A = \begin{bmatrix} \frac{1-\alpha+\beta}{1-\alpha} \Sigma_1 & -\frac{1-\alpha+\beta}{1-\alpha} \Sigma_1 + \frac{\beta \rho - (1-\alpha+\beta)(1-\gamma)\nu\sigma}{(1-\eta)\sigma} \\ \frac{\beta}{\alpha(1-\alpha)} \Sigma_1 & -\frac{\beta}{1-\alpha} \Sigma_1 - \frac{(1-\gamma)(1-\alpha+\beta)\nu}{\alpha-\beta} \end{bmatrix} \quad (26)$$

$$\mathbf{M} = \frac{1}{(1-\alpha)} \begin{bmatrix} -(1-\alpha+\beta) & \alpha(1-\alpha+\beta) \\ -\frac{\beta}{\alpha} & \beta \end{bmatrix} \Sigma_2. \quad (27)$$

If $g = \tau_k$, then $\mathbf{J}^B = \mathbf{J}^A$ —if taxes on capital and labor are equal, the local dynamic properties of the system will be identical to one without a government sector.

If the eigenvalues of the Jacobian have opposite signs, competitive equilibria are locally determinate. If both eigenvalues are negative, all paths converge to the balanced growth path and equilibria are indeterminate. The trace of \mathbf{J}^B is:

$$\text{Tr} \mathbf{J}^B = \text{Tr} \mathbf{J}^A + \frac{\tau_k - g}{1 - \tau_k} \text{Tr} \mathbf{M}, \quad (28)$$

where $\text{Tr} \mathbf{J}^A = \frac{\rho - (1-\gamma)\eta\nu}{1-\eta}$ and $\text{Tr} \mathbf{M} = -\Sigma_2$. The determinant of \mathbf{J}^B is:

$$|\mathbf{J}^B| = \frac{(\beta\rho - (1-\alpha+\beta)(1-\gamma)\nu\sigma)^2}{\alpha\sigma(\alpha-\beta)(1-\alpha)(1-\eta)} \left[\frac{\Sigma_1}{\Sigma_2} - \frac{\tau_k - g}{1 - \tau_k} \right]. \quad (29)$$

Proposition 3 *All equilibria in the neighborhood of the balanced growth path are unique if and only if $\theta \in \Theta_1$ [$\theta \in \Theta_2$], and the share of physical capital is higher [lower] than the magnitude of the externality $\alpha > \beta$ [$\alpha < \beta$].*

Proof. From Proposition 1 the term in square brackets in (29) must be negative. If $\theta \in \Theta_1$ [$\theta \in \Theta_2$] $\Rightarrow \eta < 1$ [$\eta > 1$] and therefore if $\alpha > \beta$ [$\alpha < \beta$] the determinant (29) is negative. This is sufficient to ensure that the two eigenvalues of \mathbf{J}^B have opposite signs. ■

Proposition 4 *If $\theta \in \Theta_1$ [$\theta \in \Theta_2$] and the share of physical capital is lower [higher] than the magnitude of the externality $\alpha < \beta$ [$\alpha > \beta$], there will exist a continuum of equilibria in the neighborhood of the balanced growth path if and only if the ratio of government expenditure to capital taxation is sufficiently low [high] that $\frac{\text{Tr} \mathbf{J}^A}{\Sigma_2} < \frac{\tau_k - g}{1 - \tau_k}$ [$\frac{\tau_k - g}{1 - \tau_k} < \frac{\text{Tr} \mathbf{J}^A}{\Sigma_2}$].*

Proof. If $\theta \in \Theta_1$ [$\theta \in \Theta_2$] then (29) is positive if and only if $\alpha < \beta$ [$\alpha > \beta$]. If $\theta \in \Theta_1$ [$\theta \in \Theta_2$] then $\eta < 1$ [$\eta > 1$], $\rho > (1-\gamma)\nu$ [$\rho < (1-\gamma)\nu$] and $\rho > (1-\gamma)\eta\nu$ [$\rho < (1-\gamma)\eta\nu$]. Therefore $\text{Tr} \mathbf{J}^A > 0$. If $\theta \in \Theta_1 \cup \Theta_2$ and $\alpha < \beta$ [$\alpha > \beta$], then $\Sigma_2 > 0$ [$\Sigma_2 < 0$], $\text{Tr} \mathbf{M} < 0$ [$\text{Tr} \mathbf{M} > 0$], and (28) is negative if and only if $\frac{\text{Tr} \mathbf{J}^A}{\Sigma_2} < \frac{\tau_k - g}{1 - \tau_k}$ [$\frac{\tau_k - g}{1 - \tau_k} < \frac{\text{Tr} \mathbf{J}^A}{\Sigma_2}$]. If $|\mathbf{J}^B| > 0$, and $\text{Tr} \mathbf{J}^B < 0$, both eigenvalues of \mathbf{J}^B are negative. ■

For all the cases not covered by Propositions 3 and 4, the determinant and trace of the Jacobian of \mathbf{J}^B will both be positive, the eigenvalues are both positive, and the steady state is a source. The dynamics of the system are summarized in Table 1.

		$\frac{\tau_k - g}{1 - \tau_k} \leq \frac{\Sigma_1}{\Sigma_2}$	$\frac{\Sigma_1}{\Sigma_2} < \frac{\tau_k - g}{1 - \tau_k} < \frac{\text{Tr}\mathbf{J}^A}{\Sigma_2}$	$\frac{\tau_k - g}{1 - \tau_k} = \frac{\text{Tr}\mathbf{J}^A}{\Sigma_2}$	$\frac{\text{Tr}\mathbf{J}^A}{\Sigma_2} < \frac{\tau_k - g}{1 - \tau_k}$
$\alpha > \beta$	$\theta \in \Theta_1$	No Interior BGP	Saddle Path	Saddle Path	Saddle Path
	$\theta \in \Theta_2$	No Interior BGP	Indeterminate	Hopf	Unstable
$\alpha < \beta$	$\theta \in \Theta_1$	No Interior BGP	Unstable	Hopf	Indeterminate
	$\theta \in \Theta_2$	No Interior BGP	Saddle Path	Saddle Path	Saddle Path

Table 1: The dynamic structure of the model with government expenditure and distortionary taxation.

Table 1 demonstrates the two possible conditions for the emergence of indeterminacy. As in Benhabib and Perli [4], a low degree of increasing returns in conjunction with a high intertemporal elasticity of substitution ($\theta \in \Theta_2$) will imply indeterminacy—provided the tax on labor is high, relative to the tax on capital. For lower intertemporal elasticities of substitution ($\theta \in \Theta_1$), indeterminate equilibria emerge if the production function is convex in effective labor and the tax on capital income is high.⁶

To demonstrate the dynamic structure of the model, we concentrate on the more plausible of the two parameter spaces, Θ_1 , and assume that $\sigma=1.5$, $\alpha=.285$, $g=.21$, $\rho=.05$, and $\nu = .065$. If $\beta=0$ and $\gamma=0$, both per-capita human and physical capital are growing at a rate of 1% per annum and the rate of return on physical capital is 6.5%. The horizontal line $\alpha = \beta$ divides the plane in Figure 1 between sets of $\{\beta, \tau_k\}$ that make $\det \mathbf{J}^B$ negative (below) and those that make $\det \mathbf{J}^B$ positive (above), (where $\lim_{\beta \rightarrow \alpha^-} \det \mathbf{J}^B = -\infty$ and $\lim_{\beta \rightarrow \alpha^+} \det \mathbf{J}^B = +\infty$). Below this line, equilibria are unique and above the line, shifts in the tax burden from labor to capital move the system between successive areas of positive real eigenvalues, complex eigenvalues with positive real parts, complex eigenvalues with negative real parts and negative real eigenvalues. Between the two regions with complex eigenvalues there is a Hopf bifurcation—along the curve defined by $\tau_k^{Hopf} = \frac{\text{Tr}\mathbf{J}^A + g\Sigma_2}{\text{Tr}\mathbf{J}^A + \Sigma_2}$, for $\beta > \alpha$, the eigenvalues are purely imaginary. An increase in the size of the external effect in the human capital sector γ , makes the curve representing the Hopf bifurcation in Figure 1, pivot to the right—reducing the range of fiscal policies consistent with indeterminacy. Lowering the value of σ , causes the curve to pivot to the left and for sufficiently low values, it bends backwards—as in Benhabib and Perli [4], and Xie [36], higher intertemporal elasticities of

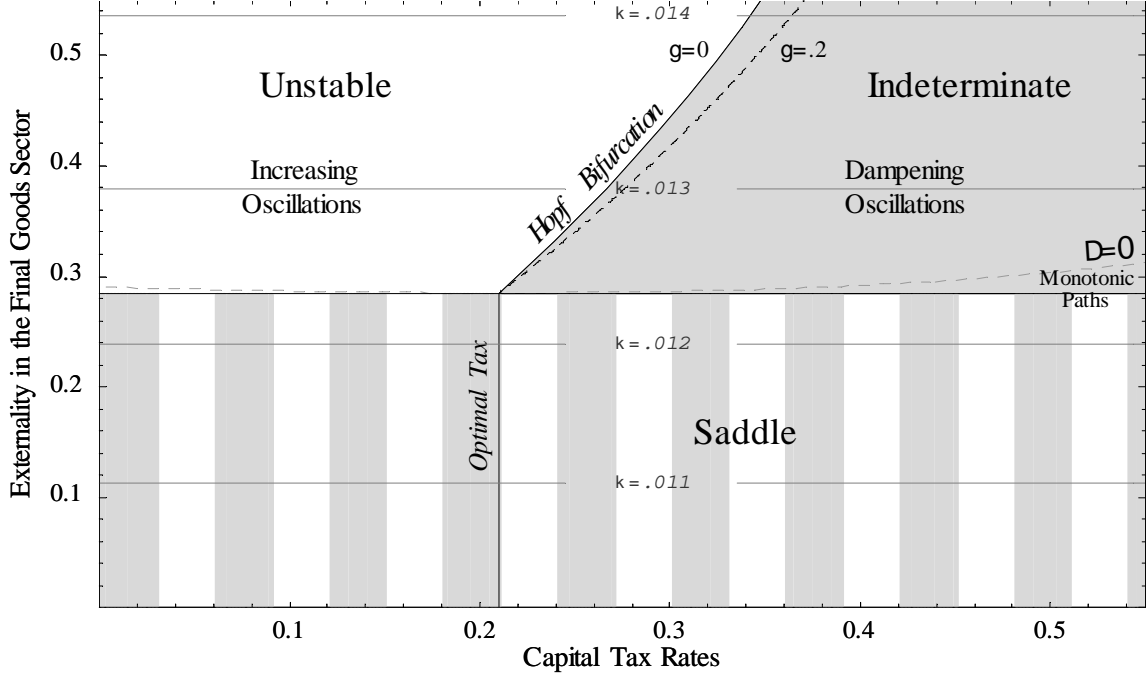


Figure 1: The dynamics of the model for various values of β and τ_k , assuming $\sigma=1.5$, $\alpha=.285$, $g = .21$, $\rho = .05$, and $\nu = .065$. The Hopf bifurcation is drawn for two values, $\gamma = 0$, and $\gamma = .2$. In the unshaded area, both eigenvalues are positive, and in the shaded both eigenvalues are negative. In the area with vertical stripes the eigenvalues have opposite signs. Above the curve that corresponds to the discriminant $\Delta = \left[\text{Tr} \left(\mathbf{J}^A + \frac{\tau_k - g}{1 - \tau_k} \mathbf{M} \right) \right]^2 - 4 \left| \mathbf{J}^A + \frac{\tau_k - g}{1 - \tau_k} \mathbf{M} \right|$, equal to zero, the eigenvalues are complex. Note that the iso-growth curves are horizontal—changes in the tax rate do not affect the long-run growth rate κ .

substitution raise the likelihood of indeterminacy.

To the right of the Hopf bifurcation in Figure 1, all balanced growth paths are stable foci—any point in their neighborhood satisfies the equilibrium conditions. As the tax burden shifts from capital to labor, the oscillations dampen more slowly, until finally the tax on capital falls below τ_k^{Hopf} , and balanced growth paths are locally unstable foci. Are there competitive equilibria in the neighborhood of an unstable focus? Yes, if the increase in the amplitude of the oscillations around it taper off—the system approaches a limit cycle, and the Hopf bifurcation is supercritical. If there are no limit cycles, the bifurcation is subcritical and the dynamic paths violate the non-negativity conditions. Our numerical simulations suggest that these Hopf bifurcations are subcritical.

With all physical capital confined to the final goods sector, indeterminacy only emerges if there are increasing returns and distortionary, non-Ramsey optimal taxation.

Proposition 5 *Indeterminacy is not possible without both increasing returns and distortionary, non-optimal taxation,*

Proof. If $\beta = 0$ then $\eta = 1 - \sigma$ and $\theta \in \Theta_1$. From Proposition 3, if $\alpha > \beta$ and $\theta \in \Theta_1$ all competitive equilibria are unique. If $\tau_k = g$, $\mathbf{J}^B = \mathbf{J}^A$ and $\text{Tr}\mathbf{J}^A > 0$. ■

The inter-sector spillovers in Lucas' version of the model play a critical role in generating the indeterminacy found by Benhabib and Perli [4]. To see why, consider what happens in Lucas' model if some people decide that in the next period, returns to physical capital will rise. Savings increase and the stock of capital accumulates at a faster rate. Because of the complementarity between capital and effective labor, wages rise—people respond by shifting from human capital production, to the production of physical goods. The increase in effective labor in the final goods sector raises the return on capital. Thus, if the initial belief—that the rate of return on physical capital will be higher—is shared by enough people, the decision to increase investment is justified, and the prediction self-fulfilling. This process is not explosive because the decline in the stock of human capital causes total factor productivity to decline.

This explains a seeming paradox—balanced growth paths in the Lucas model with its expansive type of external effects may be indeterminate (if the intertemporal elasticity of substitution is very high) or saddle-path stable, while more modest external effects imply explosive dynamics. By adding taxes to dampen physical capital accumulation, we exploited this loss of stability to generate indeterminate equilibria with empirically plausible intertemporal elasticities of substitution; though increasing returns sufficient to imply positively-sloped aggregate labor demand curves are required. Such high degrees of increasing returns are consistent with the results in Hall [17], Caballero and Lyons [7], Farmer and Guo [13], and Paul and Siegel [26]. However calculations by Basu and Fernald [1], Burnside [6], and Harrison [18] find small increasing returns, or none at all. In the remaining sections of this paper, we consider two generalizations of the model, that significantly lower, or eliminate, the degree of increasing returns necessary to generate indeterminacy.

6 Elastic Labor Supply

We introduce elastic labor supply by adding to the utility function a term for all time devoted to non-leisure activities L :

$$\max_{c,L} \int_0^\infty \left(e^{-\rho t} \frac{c^{1-\sigma}}{1-\sigma} - \frac{L^{1-\varepsilon}}{1-\varepsilon} \right) dt, \quad (\text{P.2})$$

subject to

$$\begin{aligned} \dot{k} &= (1 - \tau_l) wuh + (1 - \tau_k) rk - c \\ \dot{h} &= \nu (L - u)^{1-\gamma} h^{1-\gamma} (L_a - u_a)^\gamma h_a^\gamma. \end{aligned}$$

The term L_a represents the economy-wide average amount of time devoted to either labor market activity or human capital accumulation.

To ensure the existence of a balanced growth path, we must set $\sigma=1$ —the static elasticity of labor supply is zero and the intertemporal elasticity of labor supply is $-\frac{1}{\varepsilon}$. When labor supply is elastic, three stationary laws of motion are necessary to describe the dynamic behavior of the economy:

$$\frac{\dot{u}}{u} = \frac{\alpha}{\alpha - \beta} \left((\tau_k - g) \tilde{k}^{\alpha-1} - \frac{\tilde{c}}{\tilde{k}} \right) + \nu \frac{1 - \gamma - \alpha + \beta}{\alpha - \beta} L + \nu u \quad (30)$$

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{1}{\sigma} \left((1 - \tau_k) \alpha \tilde{k}^{\alpha-1} - \rho \right) - \vartheta \left(\frac{(1 - \gamma) \nu}{\alpha} L - \frac{\tilde{c}}{\tilde{k}} + (\tau_k - g) \tilde{k}^{\alpha-1} \right) \quad (31)$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = (1 - g) \tilde{k}^{\alpha-1} - \frac{\tilde{c}}{\tilde{k}} - \vartheta \left(\frac{(1 - \gamma) \nu}{\alpha} L - \frac{\tilde{c}}{\tilde{k}} + (\tau_k - g) \tilde{k}^{\alpha-1} \right), \quad (32)$$

where:

$$L = \left[\frac{u \tilde{c} \tilde{k}^{-\alpha}}{1 - \alpha - g + \alpha \tau_k} \right]^{\frac{1}{\varepsilon}}, \quad (33)$$

and $0 < u < L$.

We express steady state consumption, capital and hours worked, in terms of the steady state rate of growth κ :

$$\tilde{c}^* = \frac{(1 - g) \rho + \kappa (1 - g - \alpha (1 - \tau_k))}{(1 - \tau_k)} \tilde{k}^*, \quad (34)$$

$$\tilde{k}^* = \left(\frac{\alpha (1 - \tau_k)}{\rho + \kappa} \right)^{\frac{1}{1-\alpha}}, \quad (35)$$

$$u^* = \frac{(1 - g - \alpha (1 - \tau_k)) (\kappa + \rho)}{(1 - g) (\kappa + \rho) - (1 - \tau_k) \alpha \kappa} \left(\frac{(1 - \alpha) (\kappa + \rho) + \beta \rho}{\nu (1 - \gamma) (1 - \alpha + \beta)} \right)^\varepsilon. \quad (36)$$

Setting (30) equal to zero, and inserting, (34), (35), and (36) yields:

$$\left(\frac{((1-g)(\kappa+\rho) - (1-\tau_k)\alpha\kappa)((1-\alpha)(\rho+\gamma\kappa) + \beta\rho)}{\nu(1-\gamma)(1-\alpha+\beta)(1-g-\alpha(1-\tau_k))(\kappa+\rho)} \right)^{\frac{1}{\epsilon}} - \frac{(1-\alpha)(\kappa+\rho) + \beta\rho}{\nu(1-\gamma)(1-\alpha+\beta)} = 0, \quad (37)$$

which reduces to (19), when $\eta = 0$ and $\epsilon \rightarrow \infty$.

Proposition 6 *A balanced growth path that corresponds to an interior solution to the agents maximization problem exists for any real $\kappa > 0$ such that the condition (37) is satisfied.*

Proof. The assumption that $\tau_l < 1$ implies that $1-g > \alpha - \alpha\tau_k$, therefore \tilde{c}^* in (34) and u^* in (36) are positive. Using (33), (34), (35), and (36) the amount of time in steady state devoted to human capital accumulation is $L^* - u^* = \frac{(1-\alpha)\kappa}{(1-\alpha+\beta)\nu} > 0$. ■

Ladron-de-Guevara *et al.* [19], demonstrate that in an undistorted endogenous growth model with a utility function that is additively non-separable in consumption and leisure, there can be two determinate, balanced growth paths. The non-linearity of (37) implies a similar result—particularly when ϵ approaches zero from below. If $\alpha = .285$, $\rho = .05$, $\nu = .065$ but $g = \tau_k = \gamma = \beta = 0$ and ϵ is no higher than $-.1925$, (an intertemporal elasticity of labor supply of at least 5.2), the model has two determinate balanced growth rates. The absence of decreasing private returns in the production of human capital is responsible for the non-concavity of the optimization problem P.2 and the subsequent non-uniqueness.

Higher tax rates on capital and bigger external effects in the final goods sector, lowers the labor supply elasticity necessary to generate multiple solutions to equation (37). Likewise, for a given combination of external effects, higher elasticities of labor supply increase the range of capital taxes consistent with multiple solutions to (37). Finally, when externalities in the human capital sector are high, multiple solutions to (37) emerge when rates of capital taxation are higher.

To determine the dynamic behavior of the model we linearize the system (30), (31), and (32), at values of κ that solve (37). The Jacobian of the linearized system \mathbf{J}^C , is a three dimensional, square matrix (it can be expressed as the weighted sum of three matrices: $\mathbf{J}^C = \mathbf{A} + \frac{\tau_k - g}{1 - \tau_k} \mathbf{B} + \frac{1}{\epsilon} \mathbf{C}$). Define $R(\mathbf{J}^C) = j_{12}^C j_{21}^C + j_{13}^C j_{31}^C + j_{23}^C j_{32}^C - j_{11}^C j_{22}^C - j_{11}^C j_{33}^C - j_{22}^C j_{33}^C$ —the signs of $\text{Tr}(\mathbf{J}^C)$, $R(\mathbf{J}^C)$, $|\mathbf{J}^C|$, and the term $|\mathbf{J}^C| - \text{Tr}(\mathbf{J}^C) R(\mathbf{J}^C)$, provide all the information necessary to describe the dynamic behavior of the linearized system.

In Figure 2 we illustrate the dynamic behavior of the model—we once more set $\sigma=1.5$, $\alpha=.285$, $g=.21$, $\rho=.05$ and $\nu=.065$, assume that $\epsilon=-.25$, and to ensure that P.2 is concave,

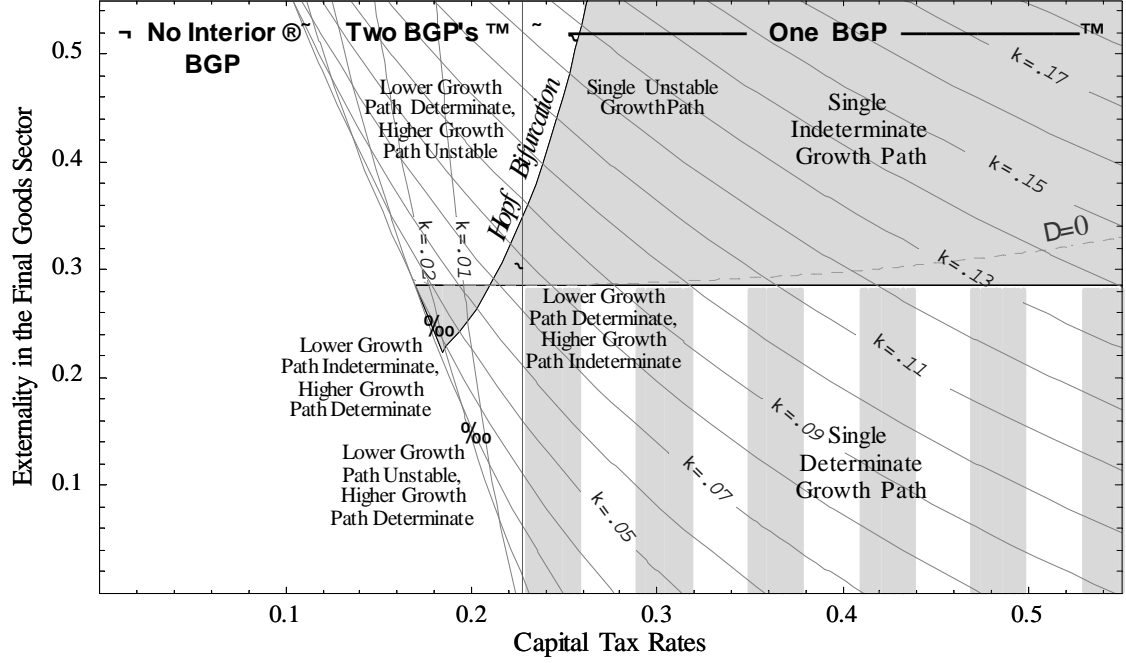


Figure 2: The dynamics of the model with elastic labor supply for various values of β and τ_k , assuming $-\frac{1}{\varepsilon}=4$, $\alpha=.285$, $g=.21$, $\rho=.05$, and $\nu=.065$. In the grey areas, equilibria are indeterminate—the lower left-hand grey triangle is a region with low growth and indeterminate equilibria. For a given tax rate the discriminant $\Delta = \frac{1}{108}(4R(\mathbf{J}^C)^3 + 4\text{Tr}(\mathbf{J}^C)^3|\mathbf{J}^C| - \text{Tr}(\mathbf{J}^C)^2 R(\mathbf{J}^C)^2 - 18\text{Tr}(\mathbf{J}^C)^2 R(\mathbf{J}^C)|\mathbf{J}^C| + 27|\mathbf{J}^C|^2)$, is equal to zero for two different values of β on the upper (higher growth) portion of the manifold (42)—above this curve eigenvalues are complex. The iso-growth curves are now diagonal—higher taxes on capital imply a higher long-run growth rate.

set γ to be a small but positive number, .001.

With the addition of elastic labor supply, the iso-growth curves in Figure 2 are no longer horizontal—fiscal policy affects long-run rates of growth. In the region where the curves overlap, there are multiple solutions to (37). On the lower portion of the manifold defined by (37), higher tax rates on capital correspond to lower iso-growth curves within this region. On the upper portion of the manifold, higher capital tax rates correspond to higher iso-growth curves. If the relative tax burden on capital is especially low, there is no interior solution to the agent's optimization problem—all time will be allocated to the production of physical goods. When capital income is heavily taxed, balanced growth paths are unique, the steady state growth rate is increasing in τ_k , and the dynamics of the model are identical for both endogenous and exogenous labor supply. If the production function is concave in effective labor, $|\mathbf{J}^C|$ is negative and equilibria are unique, and if the production

function is convex in effective labor, $|\mathbf{J}^C|$ is positive and equilibria will be indeterminate. A Hopf bifurcation exists on the upper part of the manifold with low capital taxation—in this region $|\mathbf{J}^C| > 0$, $\text{Tr}(\mathbf{J}^C) > 0$, and $R(\mathbf{J}^C) > 0$, there is one real positive eigenvalue, and where $|\mathbf{J}^C| - \text{Tr}(\mathbf{J}^C) R(\mathbf{J}^C) = 0$, the two complex eigenvalues are purely imaginary.

A Hopf bifurcation also exists on the lower part of the manifold defined by (37). To its left, the lower grey triangular in Figure 2, is a poverty trap—a region of low growth, and indeterminacy. If labor supply is elastic, agents can substitute between three instead of just two activities—indeterminacy is possible even with downward sloping labor demand.

$$|\mathbf{J}^C| = -\frac{(1-\alpha)(\kappa+\rho)^2((1-\alpha)\kappa+(1-\alpha+\beta)\rho)}{\alpha(\alpha-\beta)(1-\alpha+\beta)(1-\gamma)} \quad (38)$$

$$\times \left\{ \frac{((1-\alpha)\kappa+\rho)((1-\alpha)\gamma\kappa+(1-\alpha+\beta)\rho)}{(\kappa+\rho)((1-\alpha)\kappa+(1-\alpha+\beta)\rho)} + \left(\frac{\tau_k - g}{1-\tau_k} \right) \frac{(1-\alpha)\gamma\kappa+(1-\alpha+\beta)\rho}{(1-\alpha)\kappa+(1-\alpha+\beta)\rho} \right.$$

$$\left. + \left(-\frac{1}{\epsilon} \right) \left(\frac{\tau_k - g}{1-\tau_k} \right) \gamma + \left(-\frac{1}{\epsilon} \right) \frac{(1-\alpha)^2(\kappa+2\rho)\gamma\kappa+((1-\alpha)(\gamma-\alpha)-\alpha\beta)\rho^2}{(1-\alpha)(\kappa+\rho)^2} \right\}.$$

What are some observable conditions associated with poverty traps? A necessary (but not sufficient) condition for indeterminacy, is that $|\mathbf{J}^C| > 0$ (from the Routh-Hurwitz Theorem). Since the first of the four terms inside the brackets of (38) is positive, indeterminacy is only consistent with $\alpha > \beta$ if the sum of the remaining three terms is negative and large enough (in absolute value) to ensure that the entire term in brackets is negative as well. The last term inside the brackets is negative if $\beta > \frac{(1-\alpha)((1-\alpha)(\kappa+2\rho)\gamma\kappa-(\alpha-\gamma)\rho^2)}{\alpha\rho^2}$. Thus, for a given rate of growth, this last term is a product of the elasticity of labor supply $(-\frac{1}{\epsilon})$ and a term directly related to the elasticity of labor demand. The poverty trap in Figure 2 emerges in an economy where both labor supply and demand are highly elastic and income from capital is not too heavily taxed.

If labor supply is elastic, fiscal policy determines the steady state growth rate. Furthermore, a shift in the tax burden from labor to capital can move the economy out of a poverty trap, to a region where balance growth paths on the lower portion of the manifold defined by (37) are unstable. The economy jumps to the higher, determinate balanced growth path. Further shifts in the tax burden move the economy to the region where balanced growth paths and equilibria are unique and growth is even higher.

7 Physical Capital in the Human Capital Sector

So far we have assumed that human capital is produced with only human capital. Although there are external effects in both sectors, indeterminacy emerges when distortionary tax-

ation is combined with externalities and increasing returns, on the production side of the economy. Social returns in the human capital sector are constant, and external effects there, play only a minor role in determining the dynamic behavior of the model.

If physical capital is employed in the production of human capital, agents respond to anticipated changes to rates of return by reallocating not only human capital, but also physical capital between the two sectors. In this section we demonstrate that when the tax burden on capital is sufficiently high, indeterminacy emerges when social returns to scale are constant, and only a small degree of external effects are present in the human capital producing sector.

The maximization problem of the representative agent will be:

$$\max_c \int_0^\infty e^{-\rho t} \frac{c^{1-\sigma}}{1-\sigma} dt, \quad (\text{P.3})$$

subject to

$$\begin{aligned} \dot{k} &= (1 - \tau_l) w u h + (1 - \tau_k) r s k - c \\ \dot{h} &= \nu (1 - s)^\delta k^\delta (1 - u)^{1-\delta-\gamma} h^{1-\delta-\gamma} (1 - u_a)^\gamma h_a^\gamma, \end{aligned}$$

where s is the fraction of capital employed in the final goods sector, and δ regulates the share of physical capital in the production of human capital. We assume constant social and private returns in the final goods sector:

$$\dot{k} = (1 - g) (s k)^\alpha (u h)^{1-\alpha} - c. \quad (39)$$

We define two factor intensities— $\tilde{x}_y = \frac{s k}{u h}$ is the portion of physical capital employed in the final goods sector, divided by the portion of human capital employed in that sector, and $\tilde{x}_h = \frac{(1-s)k}{(1-u)h}$ is the analogous ratio for the human capital sector. The dynamic behavior of the economy is described by the laws of motion for detrended consumption $\tilde{c} = \frac{c}{h}$, physical capital $\tilde{k} = \frac{k}{h}$, and the ratio between the costates for human and physical capital $p = \frac{\mu}{\lambda}$:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{1}{\sigma} ((1 - \tau_k) r - \rho) - \nu (1 - u) \tilde{x}_h^\delta \quad (40)$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = (1 - g) \tilde{x}_y^\alpha u - \frac{\tilde{c}}{\tilde{k}} - \nu (1 - u) \tilde{x}_h^\delta \quad (41)$$

$$\frac{\dot{p}}{p} = -\frac{1 - g - \alpha (1 - \tau_k) w}{1 - \alpha} \frac{w}{p} + (1 - \tau_k) r. \quad (42)$$

Equilibrium factor returns, hours worked in the production sector, and factor shares are:

$$r = \alpha \tilde{x}_y^{\alpha-1} \quad (43)$$

$$w = (1 - \alpha) \tilde{x}_y^\alpha, \quad (44)$$

$$u = \frac{\tilde{k} - \tilde{x}_h}{\tilde{x}_y - \tilde{x}_h}. \quad (45)$$

$$\tilde{x}_y = \left[\left(\frac{1 - \gamma - \delta}{1 - g - \alpha(1 - \tau_k)} \right)^{1-\delta} p\nu \left(\frac{\delta}{\alpha(1 - \tau_k)} \right)^\alpha \right]^{\frac{1}{\alpha-\delta}} \quad (46)$$

$$\tilde{x}_h = \left[\left(\frac{\delta}{\alpha(1 - \tau_k)} \right)^\alpha \left(\frac{1 - \gamma - \delta}{1 - g - \alpha(1 - \tau_k)} \right)^{1-\alpha} \frac{1}{p\nu} \right]^{\frac{1}{\alpha-\delta}}. \quad (47)$$

Both external effects and taxes affect the relative factor intensities of the sectors:

$$\frac{\tilde{x}_y}{\tilde{x}_h} = \frac{(1 - \gamma - \delta) \alpha (1 - \tau_k)}{\delta (1 - g - \alpha (1 - \tau_k))}. \quad (48)$$

Proposition 7 *The final goods sector is relatively intensive in physical capital ($\tilde{x}_y > \tilde{x}_h$), if and only if $\frac{\tau_k - g}{1 - \tau_k} < \frac{\alpha(1 - \gamma) - \delta}{\delta}$ and the human capital sector is relatively intensive in physical capital ($\tilde{x}_y < \tilde{x}_h$), if and only if $\frac{\tau_k - g}{1 - \tau_k} > \frac{\alpha(1 - \gamma) - \delta}{\delta}$.*

Proof. Follows directly from (48). ■

In Bond *et al.* [5] and Mino [24], indeterminacy emerges when the final goods sector is relatively intensive in physical capital at the social level, but intensive in human capital at the private level. The same is true in this model—the former condition is satisfied if $\alpha > \delta$, and the latter if $\frac{\delta}{1 - \delta - \gamma}$ is larger than $\frac{\alpha}{1 - \alpha} \frac{1 - \tau_k}{1 - \tau_l}$. Inserting $\tau_l = \frac{g - \alpha \tau_k}{1 - \alpha}$, the condition at the private level is $\frac{\tau_k - g}{1 - \tau_k} > \frac{\alpha(1 - \gamma) - \delta}{\delta}$ —once more dynamic behavior is determined by the value of the familiar ratio $\frac{\tau_k - g}{1 - \tau_k}$.

Proposition 8 *If $\alpha > \delta$, all equilibria in the neighborhood of the balanced growth path are unique if and only if capital taxation relative to government expenditure is sufficiently low that $\frac{\tau_k - g}{1 - \tau_k} < \frac{\alpha(1 - \gamma) - \delta}{\delta}$. If $\frac{\tau_k - g}{1 - \tau_k} > \frac{\alpha(1 - \gamma) - \delta}{\delta}$ there exists a continuum of equilibria in the neighborhood of the balanced growth path. If $\alpha < \delta$ all equilibria in the neighborhood of the balanced growth path are unique if and only if $\frac{\tau_k - g}{1 - \tau_k} > \frac{\alpha(1 - \gamma) - \delta}{\delta}$ and unstable if $\frac{\tau_k - g}{1 - \tau_k} < \frac{\alpha(1 - \gamma) - \delta}{\delta}$.*

Proof. We linearize (40), (41), and (42) and denote the Jacobian as \mathbf{J}^D . Since $j_{31}^D = j_{32}^D = 0$, one eigenvalue of \mathbf{J}^D is $j_{33}^D = -\frac{(1 - \alpha + \delta)}{(\alpha - \delta)} \left[\frac{[\nu \delta^\delta (1 - \gamma - \delta)^{(1 - \delta)}]^{2 - \alpha}}{((1 - \alpha)(1 - \tau_l))^{(1 - \delta)} (\alpha(1 - \tau_k))^\delta} \right]^{\frac{1}{1 - \alpha + \delta}}$, which is negative if $\alpha > \delta$, and positive if $\alpha < \delta$. The signs of the remaining two eigenvalues are determined by the signs of the determinant and trace of the submatrix $\begin{pmatrix} j_{11}^D & j_{12}^D \\ j_{21}^D & j_{22}^D \end{pmatrix}$. Since $j_{11}^D = 0$, the trace of the submatrix is $j_{22}^D = \frac{\tilde{c}^* (\tilde{x}_y^* - \tilde{x}_h^*) + (1 - g) \tilde{x}_h^* \tilde{x}_y^{*\alpha} + \nu \tilde{x}_h^{*\delta} \tilde{k}^{*2}}{\tilde{k}^{*2} (\tilde{x}_y^* - \tilde{x}_h^*)}$. From Proposition

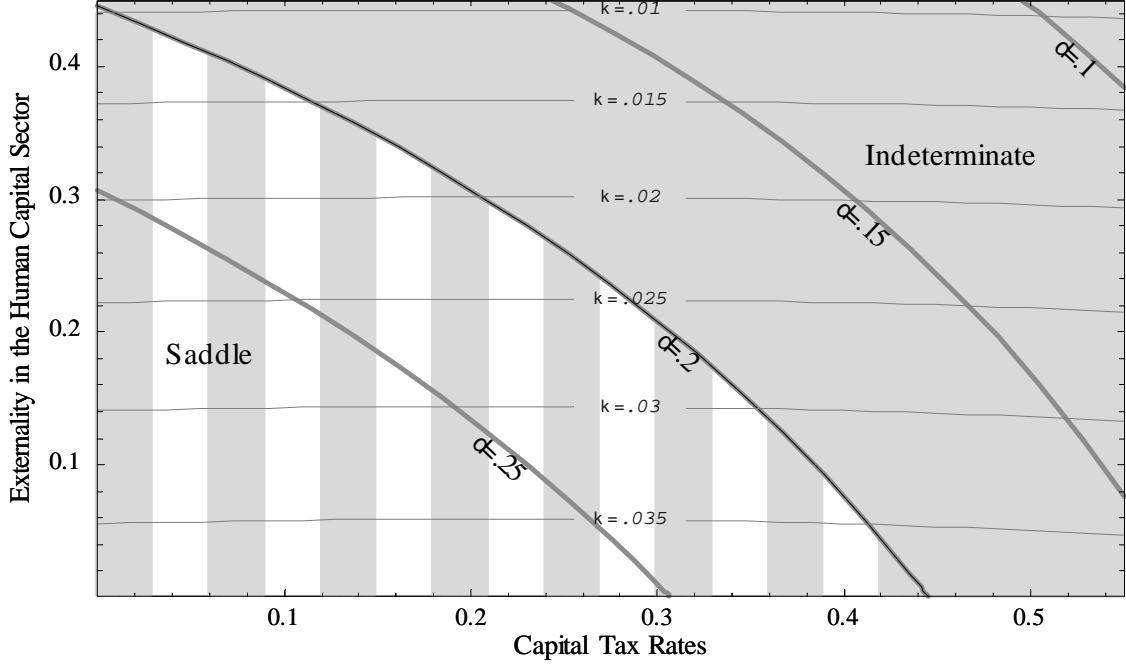


Figure 3: The dynamics of the model with physical capital employed in the human capital sector, for various values of γ and τ_k , $\alpha=.285$, $\delta = .2$, $g = .21$, $\rho =.05$, and $\nu =.12$. In the area with vertical stripes there are two positive and one negative eigenvalue. In the shaded area there are two negative and one positive eigenvalue.

7, $\tilde{x}_y > \tilde{x}_h$ if and only if $\frac{1-g}{1-\tau_k} < (1-\gamma) \frac{\alpha}{\delta}$. Therefore if $\frac{1-g}{1-\tau_k} < (1-\gamma) \frac{\alpha}{\delta}$, j_{22}^D is positive. From (45), if $\tilde{x}_y < \tilde{x}_h$ then $\tilde{k} < \tilde{x}_y$ and if $\frac{\tau_k-g}{1-\tau_k} > \frac{\alpha(1-\gamma)-\delta}{\delta}$, $j_{22}^D = \frac{(1-g)\tilde{x}_y^{\alpha} + \nu\tilde{x}_h^{*\delta}(2\tilde{k}^* - \tilde{x}_y^*)}{\tilde{k}^*(\tilde{x}_y^* - \tilde{x}_h^*)}$ is negative. The determinant of the submatrix is $-j_{21}^D j_{12}^D = -\frac{1}{\tilde{k}^*} \frac{\nu\tilde{x}_h^{*\delta}}{\tilde{x}_y^* - \tilde{x}_h^*}$, which is positive if $\frac{\tau_k-g}{1-\tau_k} < \frac{\alpha(1-\gamma)-\delta}{\delta}$, and negative if $\frac{\tau_k-g}{1-\tau_k} > \frac{\alpha(1-\gamma)-\delta}{\delta}$ —the submatrix has eigenvalues with opposite signs if $\frac{\tau_k-g}{1-\tau_k} > \frac{\alpha(1-\gamma)-\delta}{\delta}$ and two positive eigenvalues if $\frac{\tau_k-g}{1-\tau_k} < \frac{\alpha(1-\gamma)-\delta}{\delta}$. ■

Captions for Figures:

The combinations of capital taxes and external effects that support indeterminacy are presented in Figure 3 along with the iso-growth curves that correspond to solutions for (49) with $\delta = .2$. The additional curves show the borders between determinacy and indeterminacy for alternative values of δ . Indeterminate equilibria emerge for lower combinations of capital taxes, and external effects, as the value of δ rises. Proposition 8 combines the results in Bond *et al.* [5] with Mino [24]. The horizontal axis in Figure 3 corresponds to the results in the former, and the vertical axis corresponds to the results in a simplified version of the latter.⁷

If physical capital is an input in the production of human capital, fiscal policy affects the rate of steady state growth:

$$\kappa = \frac{1}{\sigma} \left[\nu (1 - \gamma - \delta)^{1-\delta} \delta^\delta (1 - g - \alpha (1 - \tau_k))^\delta \right]^{\frac{1-\alpha}{1-\alpha+\delta}} (\alpha (1 - \tau_k))^{\frac{\alpha\delta}{1-\alpha+\delta}} - \frac{\rho}{\sigma}. \quad (49)$$

Nonetheless, the relative flatness of the iso-growth curves in Figure 3 confirms the observation by Stokey and Rebelo [31], that in the absence of elastic labor supply, the ability of fiscal policy to affect growth is very modest. However, if the share of physical capital in the human capital sector is sufficiently large, fiscal policy can determine the stability of the economy—high taxes on capital income will coincide with multiple equilibria, particularly if there are some diminishing private returns in the human capital sector, or the share of physical capital in that sector is significant.

8 Conclusion

In his 1988 paper, Lucas emphasized the important steady state growth properties of the two-sector model but conceded: “The dynamics of this system are not as well understood as those of the one-good model,...” Benhabib and Perli [4], Bond *et al.* [5], Xie [36], Ladron de Guevara *et al.* [19], and the results in this paper, demonstrate that neither the uniqueness of equilibria, or even the uniqueness of balanced growth paths is robust to a variety of modest extensions that feature prominently in one-sector models with exogenous growth.

Grandmont *et al.* [14], and Pintus *et al.* [28] have analyzed the global dynamic properties of discrete-time one-sector models, particularly in the regions surrounding Hopf bifurcations. Guo and Lansing [16] and Coury and Wen [11] demonstrate that even in the region where steady states have a saddle structure, global indeterminacy may exist. By contrast, whereas the local dynamic properties of endogenous growth models with human capital accumulation are now better understood, the global dynamics of these models remain *terra incognita*. This is only one of several aspects of this model that need further investigation.

Because of their saddle structure, standard real business cycle models generate monotonic impulse responses. By contrast, models with indeterminacy have complex eigenvalues, and generate impulse responses from simple shocks that replicate many of the cyclical patterns observed in U.S. data. Indeed, from work by Benhabib and Farmer [2], Farmer and Guo [12], [13], Perli [27], Weder [34], and Wen [35], we learn that exogenous growth models with indeterminacy generate artificial time series that mimic the dynamic behavior of the U.S. economy.

Traditional real business cycle models have weak endogenous propagation mechanisms—they have trouble replicating observed autocorrelations, or impulse responses not incorporated within the dynamics of the impulses themselves (see Cogley and Nason [9]). These models also fail to mimic the shape of the spectrum of output—they neither capture its low frequency properties or generate a peak in its spectrum at business cycle frequencies. Collard [10] shows that a one-sector endogenous growth model generates a non-zero-valued spectrum for output at the zero frequency and output series with positive serial correlation. Unlike models with exogenous growth, Collard’s model produces the hump-shaped pattern of impulse responses to transitory shocks also generated by VAR estimates of the U.S. economy.

Could a model that combines endogenous growth with indeterminacy produce even better responses to shocks? And what kind of shocks? McGrattan [21] demonstrates that if the standard RBC model is altered to include distortionary taxation, fiscal policy shocks can explain more than half the fluctuations in output. Perhaps better results can be achieved by a model that combines fiscal policy with endogenous growth, and indeterminacy.

Notes

² In Lucas' version of the model $u_a = 1$ and $\gamma = 0$.

³ Schmitt-Grohé and Uribe [32] demonstrate that if government expenditure is not too pro-cyclical, a wide range of plausible fiscal policies are consistent with indeterminacy in a detrended one sector real business cycle model. In a model with endogenous growth, the common trend between output and government expenditure cannot be ignored.

⁴ Integrating the individual's budget constraint over the time variable q starting at time t , and using the first order conditions, replacing the term $h(q)$ with $h(0)e^{\int_0^q (\nu(1-u(s)))ds}$ and $(1 - \tau_l(q))w(q)$ with $e^{\int_t^q (1-\tau_k(s))r_k(s)ds} (1 - \tau_l(t))w(t)e^{\nu(t-q)}$:

$$\underbrace{\int_t^\infty e^{-\int_t^q (1-\tau_k(s))r_k(s)ds} c(q) dq}_{W} = (1 - \tau_l(t)) w(t) h(0) e^{\int_0^t (\nu(1-u(s)))ds} \underbrace{\int_t^\infty e^{-\nu \int_t^q u(s)ds} u(q) dq}_{U} + k(t)$$

The term W is equal to the present value of human wealth at time t . The value of hours devoted to physical production that maximizes human wealth from time t forward is the set of $u(s)$ that maximizes the term U . Neither the value of this term, nor the present value of consumption, is directly effected by the future time path of τ_l .

⁵ The second derivative with respect to consumption is negative:

$$\frac{\partial^2 H^0}{\partial c \partial c} = -\sigma \left[\tilde{c} \phi(t)^{\frac{1-\alpha+\beta}{1-\alpha}} \right]^{-\sigma-1}.$$

Differentiating H^0 twice with respect to capital and inserting (20):

$$\frac{\partial^2 H^0}{\partial k \partial k} = \frac{\alpha}{\tilde{k}^2} \left(\frac{1}{\sigma} \frac{\tilde{c}}{\tilde{k}} \left[(2 - \alpha)(1 - \alpha) \frac{(1 - g) \tilde{k}^{\alpha-1}}{\phi(t)^{\frac{1-\alpha+\beta}{1-\alpha}}} - 2\tilde{c} \right] \tilde{\omega} - \tilde{\zeta} (1 - \alpha)(1 - g) \tilde{k}^\alpha \right),$$

The term in square brackets approaches zero from above as $t \rightarrow +\infty$. All the other terms are negative.

⁶ Mendoza, Razin and Tesar [22] find that in the United States, between 1965 and 1988, the average tax on capital was .43, labor .25, and consumption .06. Although their calculation of labor taxation is not net of transfer payments, U.S. fiscal policy, as well as fiscal policy in the United Kingdom, Japan and Canada fall within the range consistent with indeterminacy.

⁷ Raurich [29] demonstrates that if factor returns are taxed in both sectors, and lump-sum taxation is also present, indeterminacy emerges when the tax imposed on both factors in the final goods sector, finances a flat-rate subsidy to the factors employed in the human capital sector.

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